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LETTER TO THE EDITOR

Intensity fluctuations from a one-dimensional random wavefront

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Abstract. A wave with an irregular wavefront develops intensity fluctuations through natural focusing as it propagates in free space. The moments $\langle I^m \rangle$ of the intensity distribution are calculated in terms of the wavefront randomness for a one-dimensional wavefront, in the short wavelength limit.

It is of interest in several different contexts to understand the fluctuations of intensity which develop in the free propagation of a monochromatic wave which has, initially, irregularity only in its phase—that is, a random wavefront. Twinkling starlight, for example, is produced by the random wavefront emerging from the upper atmosphere, disturbed by passage through its irregular refractive index. Similar twinkling phenomena occur in radio astronomy and ocean acoustics.

Here I shall show how the observed moments of intensity $\langle I^m \rangle$ (wavefront ensemble average) are related to the randomness of the wavefront for the simplified case of a corrugated or essentially one-dimensional random wavefront. As the term wavefront suggests, it is the geometrical optics, short wavelength, limit which will be considered, and the results will be, in this limit, exact. This important extreme of the long-standing general problem of intensity fluctuations from a random phase wave had admitted only partial solution (Shishov 1971, Buckley 1971, who obtained $\langle I^2 \rangle$) until Berry (1977) pointed out that the moments $\langle I^m \rangle$ in the geometrical limit are determined by the natural focusing which produces caustics, that is, by the hierarchy of optical catastrophes (Berry and Upstill 1980, Berry 1976, Arnold 1972). As the wavelength λ is reduced, the focusing becomes sharper—the diffraction becomes finer—and the intensity moments increase. From the geometry of catastrophes alone, Berry deduced the different asymptotic wavelength dependences of the moments: $\langle I^m \rangle \propto k^{\nu_m}$, where $k = 2\pi/\lambda$ and the *twinkling exponents* ν_m are universal rational numbers, independent of the wavefront randomness which enters only in the constant of proportionality to be calculated here. In spite of the fundamental conceptual role played by caustics and catastrophes in this theory, it is actually possible for the case of the one-dimensional wavefront to derive the central result without their explicit support, and this is the course to be adopted here, a complete account being given elsewhere (Hannay 1982).

Consider a one-dimensional irregular wavefront propagating in two-dimensional free space and described by the distance function $S(x)$ (figure 1) to a fixed observation

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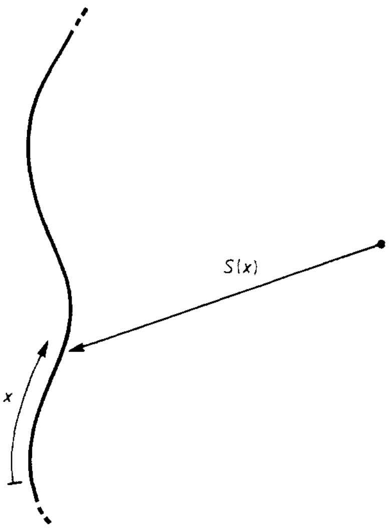

Figure 1. Irregular wavefront described by its distance $S(x)$ from a fixed observation point as a function of arc length x.

point. We shall take the intensity on the wavefront to be everywhere unity, the generalisation to non-uniform or random intensity being straightforward. Typically there will be one or more points on the wavefront where $\mathrm{d}S(x)/\mathrm{d}x = 0$, that is, where the wave normal, or ray, passes through the observation point. In the short wavelength limit it is the behaviour of the function $S(x)$ in the neighbourhood of these points which determines the observed intensity. If, for example, the distance function $S(x)$ varies only slowly in the neighbourhood, then the wavefront is rather well 'focused' on the observation point and the intensity contribution from that point is high, and limited ultimately by diffraction.

It is the occurrence of such focusings which determines the intensity moments $\langle I^m \rangle$ in the short wavelength limit. They can be classified according to the number of consecutive derivatives in the local Taylor expansion of S which vanish: the lowest-order focus is one in which just the second derivative $\mathrm{d}^2S/\mathrm{d}x^2$ is zero, but if all the higher ones up to and including the nth, but not the $(n+1)$th, are zero too, we shall say that we have an nth-order focus (or catastrophe A_n; for example, at a third-order focus there is a catastrophe A_3—the 'cusp'). If the joint probability density of the first l derivatives of S at position x (as well as the zeroth) is denoted by the function $P_x^{(l)}(S_l, S_{l-1}, \ldots, S_1, S_0)$, where $S_j = \mathrm{d}^jS(x)/\mathrm{d}x^j$, then we may introduce an n-focus probability:

$$P_x^{(n+1)}(S_{n+1}, 0, 0, \ldots, S_0)|S_{n+1}|\, \mathrm{d}S_{n+1}\, \mathrm{d}S_{n-1}\, \mathrm{d}S_{n-2} \ldots \mathrm{d}S_0\, \mathrm{d}x. \qquad (1)$$

In words, this is the joint probability that there is a zero of the function $S_n(x)$ in the interval $\mathrm{d}x$ about x, and that the *values* of the lower derivatives there lie in the respective intervals $\mathrm{d}S_{n-1}, \mathrm{d}S_{n-2}, \ldots, \mathrm{d}S_1$ about the value zero, while the zeroth and $(n+1)$th derivatives lie in the ranges $\mathrm{d}S_0$ and $\mathrm{d}S_{n+1}$ about the values S_0 and S_{n+1}. The reason for this definition will become clear in the next paragraphs.

Next we need the observed field ψ due to an n-focus. We must not, however, restrict attention to a precise n-focus because in the $k \to \infty$ limit the slightest deviation from a perfect n-focus will change ψ dramatically. (In contrast, the probability density P_x appropriate to an infinitesimally imperfect n-focus will still be that in (1) since P_x is, by assumption, smooth.) If a wavefront with an n-focus is slightly disturbed in a general way, the first n derivatives of its distance function are no longer zero. It is appropriate (for the purpose of applying the probability distributions just constructed) to consider the Taylor expansion about the position where $S_n(x)=0$. This varies with the disturbance, but it is guaranteed to exist since the disturbance is slight, and $S_{n+1}\neq 0$ by construction. (Indeed, given the disturbed wavefront alone, it is the only non-arbitrary point about which to expand.) With X measuring distance from this point the expansion is

$$S_{n+1}\frac{X^{n+1}}{(n+1)!}+S_{n-1}\frac{X^{n-1}}{(n-1)!}+S_{n-2}\frac{X^{n-2}}{(n-2)!}+\ldots+S_1X+S_0. \tag{2}$$

Here, as for a perfect n-focus, the term X^{n+1}, whose coefficient is by definition non-zero, dominates all higher-order terms which can therefore be disregarded. (Catastrophe theory justifies this properly.) The consequent wavefield in the short wavelength limit is determined by this Taylor expansion through the diffraction integral

$$\psi \underset{k\to\infty}{=} \left(\frac{k}{2\pi \mathrm{i} S_0}\right)^{1/2} \mathrm{e}^{\mathrm{i}kS_0}\int_{-\infty}^{\infty}\exp\left[\mathrm{i}k\left(S_{n+1}\frac{X^{n+1}}{(n+1)!}+S_{n-1}\frac{X^{n-1}}{(n-1)!}+\ldots+S_1X\right)\right]\mathrm{d}X. \tag{3}$$

(The symbol $\underset{k\to\infty}{=}$ is to mean that the *ratio* of the two sides tends to unity as $k\to\infty$.) As usual with asymptotic integrals (stationary phase integrals for example), the local $(X\sim 0)$ form of the integrand can safely be extended out to $\pm\infty$ because both the true integral and the approximating one have negligible contributions from the wings ($|X|$ large) where the integrands differ—hence the infinite limits in (3). The integral falls to zero for large positive or negative values of $S_1,\ldots,S_{n-1}$ because the integrand then has fine oscillations and narrow stationary regions (as measured, say, by the range of X for which the phase lies within 2π of its stationary value). The rate at which it falls obviously increases with k so that as $k\to\infty$ the field ψ drops to zero even for infinitesimal disturbances of $S_{n-1},\ldots,S_1$. The focus has become infinitely sharp as expected in the geometrical limit.

By now raising $I\equiv|\psi|^2$ to the mth power, multiplying by the probability (1), and integrating over all the variables whose differentials appear there, we obtain the preliminary result; the contribution, $\langle I^m\rangle_n$ say, of n-focusing to $\langle I^m\rangle$:

$$\begin{aligned}\langle I^m\rangle_n &\underset{k\to\infty}{=} \int_{-\infty}^{\infty}\mathrm{d}x\int_{-\infty}^{\infty}\mathrm{d}S_{n+1}\int_{-\infty}^{\infty}\mathrm{d}S_0\,P_x^{(n+1)}(S_{n+1},0,0,\ldots,0,S_0)\\ &\quad\times|S_{n+1}|\left(\int_{-\infty}^{\infty}\cdots\int|\psi|^{2m}\,\mathrm{d}S_{n-1}\ldots\mathrm{d}S_1\right)\\ &=\mathcal{N}k^{(n-1)(2m-n-2)/2(n+1)}\iiint_{-\infty}^{\infty}S_0^{-m}|S_{n+1}|^{1-(4m-n^2+n)/2(n+1)}\\ &\quad\times P_x^{(n+1)}(S_{n+1},0,0,\ldots,0,S_0)\,\mathrm{d}S_0\,\mathrm{d}S_{n+1}\,\mathrm{d}x\end{aligned} \tag{4}$$

where $\mathcal{N}$ is a numerical constant depending only on m and n, which is written out below. For the present, the important constant is that derived by Berry, the k exponent, $\frac{1}{2}(n-1)(2m-n-2)/(n+1)$. For any given moment $\langle I^m\rangle$, the focus order n for which the exponent is *greatest* dominates the moment as $k \to \infty$. Thus each moment is determined by a single particular order of focusing, easily found as follows. For each focus order n the exponent is a linear function of m shown in figure 2 and the chain of uppermost line segments gives the dominant exponent—the *twinkling* exponent.

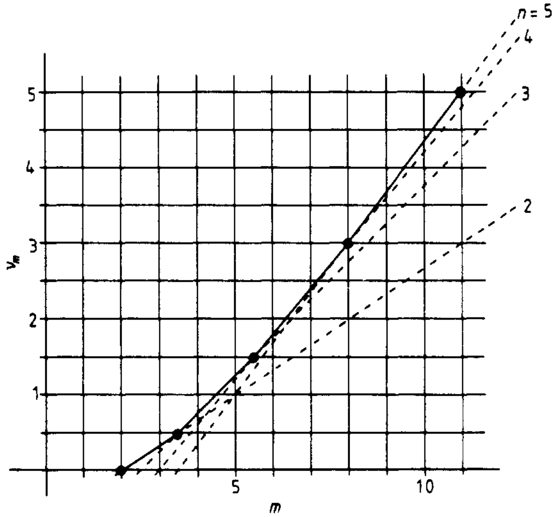

Figure 2. The twinkling exponent ν_m (full line) as a function of m (considering m as a continuous variable) constructed from the n-focus exponents (broken).

The range of m for which n is dominant is easily shown to be $\frac{1}{4}(n^2+n+2)<m<\frac{1}{4}[(n+1)^2+(n+1)+2]$ and inverting this relation we obtain the dominant focusing order

$$n = \text{integer part of } \tfrac{1}{2}[(16m-7)^{1/2}-1]. \tag{5}$$

We may expect ambiguity for moments m for which $16m-7$ is the square of an odd number so that $\frac{1}{2}[(16m-7)^{1/2}-1]$ is already an integer. Indeed, these 'exceptional' moments, which include the important moment $\langle I^2\rangle$, require a more detailed analysis of which only the results are given below. For non-exceptional moments, though, the result for $\langle I^m\rangle$ may be written down directly. All we need to do so is the assurance that interference between separate n-foci need not be accounted for (and this can be verified by showing that, for $m>2$, the probability of multiple significant n-foci occurring in the same realisation of the random wavefront vanishes as $k\to\infty$). Given this, $\langle I^m\rangle$ is simply $\langle I^m\rangle_n$ with n given by (5). In full

$$\langle I^m\rangle \underset{k\to\infty}{=} [n!^{\mu_m}(n-2)!\ldots 2!\Omega_m]k^{\nu_m} \iiint_{-\infty}^{\infty} S_0^{-m}$$

$$\times |S_{n+1}|^{1-\mu_m} P_x^{(n+1)}(S_{n+1},0,\ldots,0,S_0)\,\mathrm{d}S_0\,\mathrm{d}S_{n+1}\,\mathrm{d}x \tag{6}$$

with positive real constants μ_m, ν_m, Ω_m defined by

$$\mu_m = (4m-n^2+n)/2(n+1), \tag{7}$$

$$\nu_m = (n-1)(2m-n-2)/2(n+1), \tag{8}$$

$$\Omega_m = \int \overset{\infty}{\underset{-\infty}{\cdots}} \int \left| \frac{1}{(2\pi)^{1/2}} \int_{-\infty}^{\infty} \times \exp\left[i\left(\frac{\xi^{n+1}}{n+1} + s_{n-1}\frac{\xi^{n-1}}{n-1} + \ldots + s_1\xi \right) \right] d\xi \right|^{2m} ds_{n-1} \ldots ds_1. \tag{9}$$

This last constant Ω_m has direct catastrophe theoretical significance. The parentheses contain the catastrophe potential function of the catastrophe A_n (in the privileged 'quasihomogeneous' form where each 'control' variable s multiplies a single power of the 'state' variable ξ, thus admitting mutual scaling (Arnold 1974)). The argument of the modulus signs in (9) is then the standard form (Berry and Upstill 1980) of the characteristic diffraction pattern ('diffraction catastrophe') of A_n, and Ω_m is therefore the complete integral of the mth power of the diffraction catastrophe intensity. For example, the diffraction catastrophe for a simple caustic (A_2, the 'fold') is the Airy function, whose integral representation is seen in Ω_3. Thus

$$\Omega_3 = (2\pi)^3 \int_{-\infty}^{\infty} \mathrm{Ai}^6(x)\, dx = 9.07, \tag{10}$$

and therefore

$$\langle I^3 \rangle \underset{k\to\infty}{=} 28.8 k^{1/3} \iiint_{-\infty}^{\infty} S_0^{-3} |S_3|^{-2/3} P^{(3)}(S_3, 0, 0, S_0)\, dS_3\, dS_0\, dx. \tag{11}$$

The exceptional moments $\langle I^m \rangle$, for which $16m-7$ is the square of an odd number, require a more careful analysis because from (5) two different catastrophes, A_n and A_{n-1} where $n = \frac{1}{2}[(16m-7)^{1/2} - 1]$, are exchanging dominance. With this designation of n the result (Hannay 1982) is that (6) is modified by the substitution $n \to n-1$ within the square bracket, and multiplication by a factor $(2 \ln k)/(n+1)$ for $m \neq 2$, or three times this for $m = 2$. Thus for $m \neq 2$

$$\langle I^m \rangle \underset{k\to\infty}{=} \frac{2}{n+1} k^{\nu_m} \ln k [(n-1)!^2 (n-3)! \ldots 2! \Omega'_m] \times \iint_{-\infty}^{\infty} S_0^{-m} P_x^{(n)}(0, 0, \ldots, S_0)\, dS_0\, dx \tag{12}$$

where Ω'_m is defined by (9) with n replaced by $n-1$. The S_{n+1} in (6) has been integrated out by virtue of the identity $\mu_m = 1$, from (5), for the exceptional moments. For $m = 2$ which is especially exceptional, the result is three times this, namely

$$\langle I^2 \rangle \underset{k\to\infty}{=} 2 \ln k \iint_{-\infty}^{\infty} S_0^{-2} P_x^{(2)}(0, 0, S_0)\, dS_0\, dx. \tag{13}$$

An important sequence of specialisations is (i) to *paraxial* optics, where the wavenormals everywhere make only small angles with a principal direction of propagation, (ii) to a stationary random wavefront whose deviation from a plane is a *stationary* random function, (iii) to a stationary *Gaussian* random function, whose spectral components have independent random phases. Under these specialisations the result (13) for $\langle I^2 \rangle$ reduces to those derived by Shishov (1971) and Buckley (1971).

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